CHAPTER 6

TRIGONOMETRIC IDENTITIES AND EQUATIONS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Apply the reciprocal, quotient, and Pythagorean identities along with identities for negative angles to problem solving.

2. Apply the sum and difference, double-angle, and half-angle formulas to problem solving.

3. Apply inverse trigonometric functions to problem solving.

4. Find solutions to trigonometric equations.

INTRODUCTION

This is the final chapter dealing directly with trigonometry and trigonometric relationships. This chapter includes the basic identities, formulas for identities involving more than one angle, and formulas for identities involving multiples of an angle.

Also included in this chapter are inverse trigonometric functions and methods for solving trigonometric equations.

FUNDAMENTAL IDENTITIES

An equality that is true for all values of an unknown is called an *identity*. Many of the identities that will be considered in this section were established in earlier chapters and will be used here to change the form of an expression.

Problems in identities are often given as equalities. The identity is established by either transforming the left side into the right side or transforming the right side into the left side. Never work across the equality sign.

We have no hard-and-fast rules to use in verifying identities. However, we do offer the following suggestions:

1. *Know the basic identities given in this section.*
2. *Attempt to transform the more complicated side into the other side.*
3. *When possible, express all trigonometric functions in the equation in terms of sine and cosine.*
4. *Perform any factoring or algebraic operations.*

RECIPROCAL IDENTITIES

The *reciprocal identities* were first introduced in chapter 3. They are as follows:

$$\cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

EXAMPLE: Use the reciprocal identities to find an equivalent expression involving only sines and cosines; then simplify for

$$\frac{\sec\theta}{\csc\theta + \sec\theta}$$

SOLUTION:

$$\frac{\sec\theta}{\csc\theta + \sec\theta} = \frac{\dfrac{1}{\cos\theta}}{\dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta}}$$

$$= \frac{\dfrac{1}{\cos\theta}}{\dfrac{\cos\theta + \sin\theta}{\sin\theta\cos\theta}}$$

$$= \left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta\cos\theta}{\cos\theta + \sin\theta}\right)$$

$$= \frac{\sin\theta}{\cos\theta + \sin\theta}$$

QUOTIENT IDENTITIES

The *quotient identities* were also introduced in chapter 3.

They are as follows:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

EXAMPLE: Use the quotient identities to find an equivalent expression involving only sines and cosines; then simplify for

$$\frac{\tan \theta}{\cot \theta}$$

SOLUTION:

$$\frac{\tan \theta}{\cot \theta} = \frac{\dfrac{\sin \theta}{\cos \theta}}{\dfrac{\cos \theta}{\sin \theta}}$$

$$= \left(\frac{\sin \theta}{\cos \theta}\right)\left(\frac{\sin \theta}{\cos \theta}\right)$$

$$= \frac{\sin^2\theta}{\cos^2\theta}$$

PYTHAGOREAN IDENTITIES

Another group of fundamental identities, called the *Pythagorean identities*, involves the squares of the functions. These identities are so named because the Pythagorean theorem is used in their development.

Consider

$$x^2 + y^2 = r^2$$

and divide both sides by r^2 to get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

or

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

Since $\cos\theta = x/r$ and $\sin\theta = y/r$, then

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

or

$$\cos^2\theta + \sin^2\theta = 1$$

which is one of the Pythagorean identities.

In the same manner, dividing both sides of the equation

$$x^2 + y^2 = r^2$$

by x^2 (where $x \neq 0$) gives

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

or

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

Since $\tan\theta = y/x$ and $\sec\theta = r/x$, then

$$1 + (\tan\theta)^2 = (\sec\theta)^2$$

or

$$1 + \tan^2\theta = \sec^2\theta$$

which is another one of the Pythagorean identities.

Dividing both sides of the equation

$$x^2 + y^2 = r^2$$

by y^2 (where $y \neq 0$) gives

$$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2}$$

or

$$1 + \left(\frac{x}{y}\right)^2 = \left(\frac{r}{y}\right)^2$$

Since $\cot \theta = x/y$ and $\csc \theta = r/y$, then

$$1 + (\cot \theta)^2 = (\csc \theta)^2$$

or

$$1 + \cot^2\theta = \csc^2\theta$$

which is also one of the Pythagorean identities.

EXAMPLE: Use the Pythagorean identities to find an equivalent expression involving only sines and cosines; then simplify for

$$(\csc^2\theta - 1)(\tan^2\theta + 1)$$

SOLUTION:

$$\begin{aligned}(\csc^2\theta - 1)(\tan^2\theta + 1) &= \cot^2\theta \sec^2\theta \\ &= \left(\frac{\cos^2\theta}{\sin^2\theta}\right)\left(\frac{1}{\cos^2\theta}\right) \\ &= \frac{1}{\sin^2\theta}\end{aligned}$$

IDENTITIES FOR NEGATIVE ANGLES

The following *identities for negative angles* were first introduced in chapter 4:

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

EXAMPLE: Use the identities for negative angles to find an equivalent expression involving only sines and cosines with positive angles; then simplify for

$$\frac{\tan(-\theta)}{\cos(-\theta)}$$

SOLUTION:

$$\begin{aligned}\frac{\tan(-\theta)}{\cos(-\theta)} &= \frac{-\tan \theta}{\cos \theta} \\ &= \frac{-\dfrac{\sin \theta}{\cos \theta}}{\cos \theta} \\ &= -\frac{\sin \theta}{\cos^2\theta}\end{aligned}$$

VERIFYING TRIGONOMETRIC IDENTITIES

The process of verifying trigonometric identities is similar to simplifying trigonometric expressions except that we know in advance the desired result. Remember to use the suggestions we offered at the beginning of this chapter when verifying trigonometric identities.

EXAMPLE: Verify the identity

$$\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$$

SOLUTION:

$$\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = \frac{\sin\theta}{\dfrac{1}{\sin\theta}} + \frac{\cos\theta}{\dfrac{1}{\cos\theta}}$$

$$= (\sin\theta)(\sin\theta) + (\cos\theta)(\cos\theta)$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

EXAMPLE: Verify the identity

$$1 + \cot^2 2x = \frac{1}{\sin^2 2x}$$

SOLUTION:

$$1 + \cot^2 2x = \csc^2 2x$$

$$= \frac{1}{\sin^2 2x}$$

EXAMPLE: Verify the identity

$$2\sec\theta = \frac{\cos(-\theta)}{1 - \sin(-\theta)} + \frac{\cos(-\theta)}{1 + \sin(-\theta)}$$

SOLUTION: $\dfrac{\cos(-\theta)}{1 - \sin(-\theta)} + \dfrac{\cos(-\theta)}{1 + \sin(-\theta)}$

$$= \frac{\cos\theta}{1 + \sin\theta} + \frac{\cos\theta}{1 - \sin\theta}$$

$$= \frac{(\cos\theta)(1 - \sin\theta) + (\cos\theta)(1 + \sin\theta)}{(1 + \sin\theta)(1 - \sin\theta)}$$

$$= \frac{\cos\theta - \cos\theta\sin\theta + \cos\theta + \cos\theta\sin\theta}{1 + \sin\theta - \sin\theta - \sin^2\theta}$$

$$= \frac{2\cos\theta}{1 - \sin^2\theta}$$

$$= \frac{2\cos\theta}{\cos^2\theta}$$

$$= \frac{2}{\cos\theta}$$

$$= 2\sec\theta$$

EXAMPLE: Verify the identity

$$\frac{1 + 2\sin\theta + \sin^2\theta}{\cos^2\theta} = \frac{\sec\theta + \tan\theta}{\sec\theta - \tan\theta}$$

SOLUTION:

$$\frac{\sec\theta + \tan\theta}{\sec\theta - \tan\theta} = \frac{\dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}}$$

$$= \frac{\dfrac{1 + \sin\theta}{\cos\theta}}{\dfrac{1 - \sin\theta}{\cos\theta}}$$

$$= \left(\frac{1 + \sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{1 - \sin\theta}\right)$$

$$= \frac{1 + \sin\theta}{1 - \sin\theta}$$

$$= \left(\frac{1 + \sin\theta}{1 - \sin\theta}\right)\left(\frac{1 + \sin\theta}{1 + \sin\theta}\right)$$

$$= \frac{1 + \sin\theta + \sin\theta + \sin^2\theta}{1 - \sin\theta + \sin\theta - \sin^2\theta}$$

$$= \frac{1 + 2\sin\theta + \sin^2\theta}{1 - \sin^2\theta}$$

$$= \frac{1 + 2\sin\theta + \sin^2\theta}{\cos^2\theta}$$

PRACTICE PROBLEMS:

Verify the following identities:

1. $\dfrac{1}{\tan^2 x + 1} = \cos^2 x$

2. $\csc\theta - \sin\theta = \cos\theta\cot\theta$

3. $\dfrac{\sin^2\theta}{1 + \cos\theta} = 1 - \cos\theta$

4. $\sin^2\theta = [\cos(-\theta)][\sec(-\theta) - \cos(-\theta)]$

5. $1 - \cos^2 x = (\tan^2 x)(1 - \sin^2 x)$

6. $\dfrac{1 - \cos^2\theta}{\csc\theta} = \sin^3\theta$

7. $\dfrac{1}{2 + \cot^2(-\theta)} = \dfrac{1}{2\csc^2(-\theta) - \cot^2(-\theta)}$

NOTE: No ANSWERS are furnished since the result is known in advance for each of the preceding PRACTICE PROBLEMS.

FORMULAS FOR IDENTITIES

In this section we will discuss the trigonometric formulas for the sum and difference of angles, for double angles, and for half angles.

SUM AND DIFFERENCE FORMULAS

The fundamental identities discussed in the previous section involved functions of a single angle. In this section we will consider identities involving functions of more than one angle.

We will start by developing a formula for cos $(\alpha - \beta)$. Refer to figure 6-1. Angles α and β are constructed in standard position, so angle *KOL* is equal to α and angle *KOM* is equal to β. We will also construct angle *KON* equal to $\alpha - \beta$. Since triangles *KON* and *MOL* are similar triangles, then sides *LM* and *KN* have the same length.

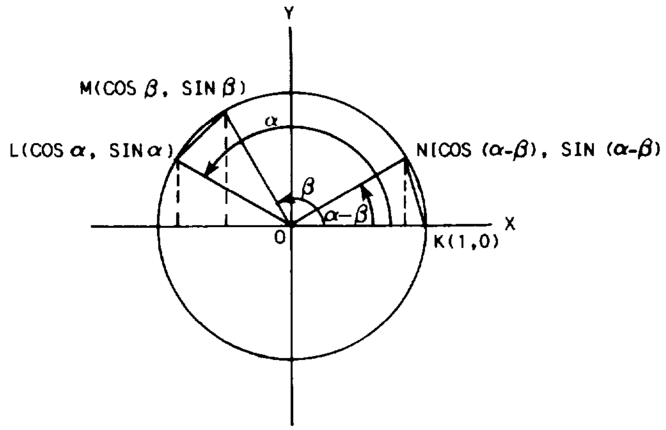

Figure 6-1.—Developing formula for cos $(\alpha - \beta)$.

Now we need to determine the coordinates of points *K, L, M,* and *N*. Recall the properties of right triangles, quadrantal angles, and reduction formulas. For the unit circle, where $r = 1$, the coordinates of point *K*, which lie on the positive *X* axis, are (1,0). According to properties of right triangles

$$\cos \theta = \frac{x}{r}$$

and

$$\sin \theta = \frac{y}{r}$$

So the coordinates of point *N* are $[\cos(\alpha - \beta), \sin(\alpha - \beta)]$.

Recall from chapter 4 that

$$\cos(180^\circ - \theta) = -\cos\theta$$

and

$$\sin(180^\circ - \theta) = \sin\theta$$

where θ is a positive acute angle. If we apply these formulas to angles α and β and note that the coordinates of a point in the second quadrant are $(-x,y)$, then the coordinates of point *L* are $(\cos\alpha, \sin\alpha)$ and the coordinates of point *M* are $(\cos\beta, \sin\beta)$.

Using the coordinates of these points and the distance formula, we can determine the lengths of sides LM and KN. Hence,

$$(LM)^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$$

$$= \cos^2\alpha - 2 \cos \alpha \cos \beta + \cos^2\beta$$

$$+ \sin^2\alpha - 2 \sin \alpha \sin \beta + \sin^2\beta$$

$$= 2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta$$

and

$$(KN)^2 = [1 - \cos (\alpha - \beta)]^2 + [0 - \sin (\alpha - \beta)]^2$$

$$= 1 - 2 \cos (\alpha - \beta) + \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)$$

$$= 2 - 2 \cos (\alpha - \beta)$$

Since sides LM and KN have the same length, we can equate the distances and simplify as follows:

$$2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta = 2 - 2 \cos (\alpha - \beta)$$

$$-2(\cos \alpha \cos \beta + \sin \alpha \sin \beta) = -2 \cos (\alpha - \beta)$$

$$\cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos (\alpha - \beta)$$

Therefore, *the cosine of the difference of two angles is equal to the cosine of the first angle times the cosine of the second angle plus the sine of the first angle times the sine of the second angle*; that is,

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

EXAMPLE: Simplify $\cos (90° - \beta)$.

SOLUTION: If

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

then

$$\cos (90° - \beta) = \cos 90° \cos \beta + \sin 90° \sin \beta$$

$$= (0)(\cos \beta) + (1)(\sin \beta)$$

$$= \sin \beta$$

which is the same result shown in chapter 4.

EXAMPLE: Determine cos 15° using the cosine of the difference of two angles.

SOLUTION:

$$\begin{aligned}\cos 15^\circ &= \cos (45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

To develop a formula for $\cos (\alpha + \beta)$, we will substitute $(-\beta)$ for β into the formula for $\cos (\alpha - \beta)$ as follows:

$$\begin{aligned}\cos (\alpha + \beta) &= \cos [\alpha - (-\beta)]\\ &= \cos \alpha \cos (-\beta) + \sin \alpha \sin (-\beta)\\ &= \cos \alpha \cos \beta - \sin \alpha \sin \beta\end{aligned}$$

Therefore, *the cosine of the sum of two angles is equal to the cosine of the first angle times the cosine of the second angle minus the sine of the first angle times the sine of the second angle;* that is,

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

EXAMPLE: Determine cos 105° using the cosine of the sum of two angles.

SOLUTION:

$$\begin{aligned}\cos 105^\circ &= \cos (45^\circ + 60^\circ)\\ &= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

We will now use the identities

$$\cos\theta = \sin(90° - \theta)$$

and

$$\sin\theta = \cos(90° - \theta)$$

and the formula

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

to develop a formula for $\sin(\alpha + \beta)$ as follows:

$$\begin{aligned}\sin(\alpha + \beta) &= \cos[90° - (\alpha + \beta)]\\ &= \cos[(90° - \alpha) - \beta]\\ &= \cos(90° - \alpha)\cos\beta + \sin(90° - \alpha)\sin\beta\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}$$

Therefore, *the sine of the sum of two angles is equal to the sine of the first angle times the cosine of the second angle plus the cosine of the first angle times the sine of the second angle;* that is,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

EXAMPLE: Verify that

$$\sin(\alpha + 45°) = \frac{\sqrt{2}}{2}(\sin\alpha + \cos\alpha)$$

SOLUTION:

$$\begin{aligned}\sin(\alpha + 45°) &= \sin\alpha\cos 45° + \cos\alpha\sin 45°\\ &= (\sin\alpha)\left(\frac{\sqrt{2}}{2}\right) + (\cos\alpha)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{2}(\sin\alpha + \cos\alpha)\end{aligned}$$

Substituting $(-\beta)$ for β into the formula for $\sin(\alpha + \beta)$ produces

$$\begin{aligned}\sin(\alpha - \beta) &= \sin[\alpha + (-\beta)]\\ &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}$$

Therefore, *the sine of the difference of two angles is equal to the sine of the first angle times the cosine of the second angle minus the cosine of the first angle times the sine of the second angle*; that is,

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

EXAMPLE: Use the formula for the sine of the difference of two angles to determine the value of

$$\sin 40^\circ \cos 10^\circ - \cos 40^\circ \sin 10^\circ$$

SOLUTION:

$$\begin{aligned}\sin 40^\circ \cos 10^\circ - \cos 40^\circ \sin 10^\circ &= \sin(40^\circ - 10^\circ)\\ &= \sin 30^\circ\\ &= 1/2\end{aligned}$$

Now, using the identity

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

and the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, we can develop a formula for $\tan(\alpha + \beta)$ as follows:

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}\\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\end{aligned}$$

Dividing both the numerator and denominator by $\cos\alpha\cos\beta$ gives

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\end{aligned}$$

Therefore, *the tangent of the sum of two angles is equal to the quantity of the tangent of the first angle plus the tangent of the* *second angle divided by the quantity of 1 minus the tangent of the first angle times the tangent of the second angle*; that is,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

EXAMPLE: If $\sin\alpha = -4/5$ and $\cos\beta = 12/13$, where α is in quadrant III and β is in quadrant IV, find $\tan(\alpha + \beta)$.

SOLUTION: Refer to figure 6-2. If $\sin\alpha = -4/5$ and α is in quadrant III, then $\tan\alpha = 4/3$. Likewise, if $\cos\beta = 12/13$ and β is in quadrant IV, then $\tan\beta = -5/12$. Therefore,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{(4/3) + (-5/12)}{1 - (4/3)(-5/12)}$$

$$= \frac{11/12}{1 + 20/36}$$

$$= \left(\frac{11}{12}\right)\left(\frac{36}{56}\right)$$

$$= \frac{33}{56}$$

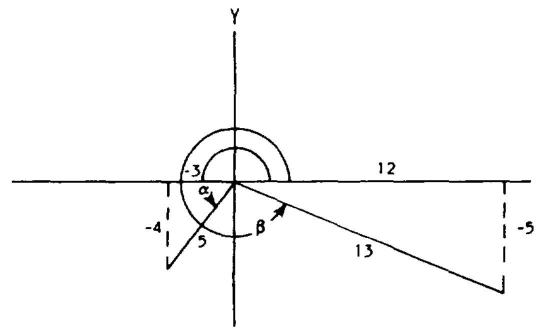

Figure 6-2.—Triangles in quadrants III and IV.

As before, to develop a formula for $\tan(\alpha - \beta)$, we will substitute $(-\beta)$ for β into the formula for $\tan(\alpha + \beta)$ as follows:

$$\tan(\alpha - \beta) = \tan[\alpha + (-\beta)]$$

$$= \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Therefore, *the tangent of the difference of two angles is equal to the quantity of the tangent of the first angle minus the tangent of the second angle divided by the quantity of 1 plus the tangent of the first angle times the tangent of the second angle*; that is,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

EXAMPLE: If $\csc \alpha = 29/21$, $\sin \beta = -8/17$, $\cos \alpha$ is negative, and $\sec \beta$ is positive, find $\cot (\alpha - \beta)$.

SOLUTION: Note that

$$\cot (\alpha - \beta) = \frac{1}{\tan (\alpha - \beta)}$$

So we determine the value of $\cot (\alpha - \beta)$ using the formula for $\tan (\alpha - \beta)$. Since $\csc \alpha$ is positive and $\cos \alpha$ is negative in quadrant II, then $\tan \alpha = -21/20$. Likewise, since $\sin \beta$ is negative and $\sec \beta$ is positive in quadrant IV, then $\tan \beta = -8/15$. Hence,

$$\begin{aligned}
\tan (\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \\
&= \frac{(-21/20) - (-8/15)}{1 + (-21/20)(-8/15)} \\
&= \frac{-31/60}{1 + 14/25} \\
&= \left(\frac{-31}{60}\right)\left(\frac{25}{39}\right) \\
&= -\frac{155}{468}
\end{aligned}$$

Therefore,

$$\cot (\alpha - \beta) = -\frac{468}{155}$$

PRACTICE PROBLEMS:

Use sum and difference formulas to find the values of the following:

1. $\sin \frac{13\pi}{12}$

2. $\cot 165°$

Verify the following using sum and difference formulas:

3. $\dfrac{\cos(\alpha - \beta)}{\cos\alpha \sin\beta} = \cot\beta + \tan\alpha$

4. $\tan\left(\alpha - \dfrac{\pi}{4}\right) = \dfrac{\tan\alpha - 1}{\tan\alpha + 1}$

5. If $\sin\alpha = -1/4$ and $\cos\beta = -4/5$, where α and β are both in quadrant III, find $\cos(\alpha + \beta)$.

ANSWERS:

1. $\dfrac{-\sqrt{6} + \sqrt{2}}{4}$

2. $\dfrac{1 + \sqrt{3}}{1 - \sqrt{3}}$ or $-2 - \sqrt{3}$

3. Result is known

4. Result is known

5. $\dfrac{4\sqrt{15} - 3}{20}$

DOUBLE-ANGLE FORMULAS

Formulas for the functions of twice an angle may be derived from the functions of the sum of two angles. Setting $\beta = \alpha$ in the formulas for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, and $\tan(\alpha + \beta)$ gives the following results:

$$\sin(\alpha + \alpha) = \sin\alpha \cos\alpha + \cos\alpha \sin\alpha$$

$$\cos(\alpha + \alpha) = \cos\alpha \cos\alpha - \sin\alpha \sin\alpha$$

$$\tan(\alpha + \alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha \tan\alpha}$$

Hence,

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 1 - 2 \sin^2\alpha \\ &= 2 \cos^2\alpha - 1\end{aligned}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2\alpha}$$

The previous formulas are known as the *double-angle formulas.*

EXAMPLE: Find the values for $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, if $\tan \theta = -12/5$ and θ is in the second quadrant.

SOLUTION: Since θ is in the second quadrant, then $\sin \theta = 12/13$ and $\cos \theta = -5/13$; so,

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta \\ &= 2\left(\frac{12}{13}\right)\left(\frac{-5}{13}\right) \\ &= -\frac{120}{169}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{-5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 \\ &= \frac{25}{169} - \frac{144}{169} \\ &= -\frac{119}{169}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2\theta} \\ &= \frac{2(-12/5)}{1 - (-12/5)^2} \\ &= \frac{-24/5}{1 - 144/25} \\ &= \frac{-24/5}{-119/25} \\ &= \left(\frac{24}{5}\right)\left(\frac{25}{119}\right) \\ &= \frac{120}{119}\end{aligned}$$

EXAMPLE: Verify that

$$\csc 2x = \frac{1}{2} \csc x \sec x$$

SOLUTION:

$$\begin{aligned}\frac{1}{2} \csc x \sec x &= \frac{1}{2}\left(\frac{1}{\sin x}\right)\left(\frac{1}{\cos x}\right)\\ &= \frac{1}{2 \sin x \cos x}\\ &= \frac{1}{\sin 2x}\\ &= \csc 2x\end{aligned}$$

HALF-ANGLE FORMULAS

From the double-angle formulas we can derive the *half-angle formulas*. Since

$$\cos 2\alpha = 1 - 2 \sin^2\alpha$$

then solving for $\sin \alpha$ results in

$$\sin \alpha = \pm \sqrt{\frac{1 - \cos 2\alpha}{2}}$$

Now, if $2\alpha = \theta$, so that $\alpha = \theta/2$, then

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

which is the half-angle formula for $\sin \theta/2$.

The half-angle formula for $\cos \theta/2$ can be obtained by solving

$$\cos 2\alpha = 2 \cos^2\alpha - 1$$

for $\cos \alpha$ or

$$\cos \alpha = \pm \sqrt{\frac{1 + \cos 2\alpha}{2}}$$

As before, if $2\alpha = \theta$, so that $\alpha = \theta/2$, then

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

The half-angle formula for tan $\theta/2$ is derived from the half-angle formulas for sine and cosine as follows:

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}}$$

$$= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

NOTE: For the half-angle formulas, the positive or negative sign is selected according to the quadrant in which $\theta/2$ lies.

EXAMPLE: Use the half-angle formulas to find the cosine, sine, and tangent of 112.5°.

SOLUTION: Since 112.5° lies in quadrant II, the cosine and tangent will be negative and the sine will be positive; so,

$$\cos 112.5° = -\sqrt{\frac{1+\cos 225°}{2}}$$

$$= -\sqrt{\frac{1+(-\sqrt{2}/2)}{2}}$$

$$= -\sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= -\frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\sin 112.5° = \sqrt{\frac{1-\cos 225°}{2}}$$

$$= \sqrt{\frac{1-(-\sqrt{2}/2)}{2}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\tan 112.5° = -\sqrt{\frac{1 - \cos 225°}{1 + \cos 225°}}$$

$$= -\sqrt{\frac{1 - (-\sqrt{2}/2)}{1 + (-\sqrt{2}/2)}}$$

$$= -\sqrt{\frac{1 + \sqrt{2}/2}{1 - \sqrt{2}/2}}$$

$$= -\sqrt{\frac{2 + \sqrt{2}}{2 - \sqrt{2}}}$$

$$= -\sqrt{\left(\frac{2 + \sqrt{2}}{2 - \sqrt{2}}\right)\left(\frac{2 + \sqrt{2}}{2 + \sqrt{2}}\right)}$$

$$= -\sqrt{\frac{6 + 4\sqrt{2}}{2}}$$

$$= -\sqrt{3 + 2\sqrt{2}}$$

EXAMPLE: Verify that

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{\sec \theta - 1}{2 \sec \theta}$$

SOLUTION:

$$\frac{\sec \theta - 1}{2 \sec \theta} = \frac{\frac{1}{\cos \theta} - 1}{2\left(\frac{1}{\cos \theta}\right)}$$

$$= \frac{\frac{1 - \cos \theta}{\cos \theta}}{\frac{2}{\cos \theta}}$$

$$= \frac{1 - \cos \theta}{2}$$

$$= \sin^2\left(\frac{\theta}{2}\right)$$

PRACTICE PROBLEMS:

1. Find the values for $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, if $\theta = \pi$.

2. Find the values for $\sin \theta/2$, $\cos \theta/2$, and $\tan \theta/2$ if $\sec \theta = 17/8$, $\tan \theta$ is positive, and $0 \leq \theta \leq 360°$.

Verify the following using double-angle and half-angle formulas:

3. $(1 + \tan x)(\tan 2x) = \dfrac{2 \tan x}{1 - \tan x}$

4. $\dfrac{2}{1 + \cos \theta} - \tan^2\left(\dfrac{\theta}{2}\right) = 1$

ANSWERS:

1. $\sin 2\theta = 0$

   $\cos 2\theta = 1$

   $\tan 2\theta = 0$

2. $\sin \theta/2 = 3\sqrt{34}/34$

   $\cos \theta/2 = 5\sqrt{34}/34$

   $\tan \theta/2 = 3/5$

3. Result is known

4. Result is known

INVERSE TRIGONOMETRIC FUNCTIONS

In this section we will discuss the notations that apply to the inverse trigonometric functions along with the principal values of the inverse functions.

NOTATION

Let us consider the inverse of the sine function, $y = \sin x$. The inverse of the sine function may be denoted as

$$x = \sin y$$

or

$$y = \sin^{-1}x$$

which can be read "the inverse sine of x." Note that $\sin^{-1}x$ does not mean $1/\sin x$.

The inverse of the sine function may also be denoted by

$$y = \arcsin x$$

which can be read "the arc sine of x." The notation $\arcsin x$ arises because it is the length of an arc on the unit circle for which the sine is x.

Similar notation occurs for the inverses of the other trigonometric functions; that is, $\cos^{-1}x$ or $\arccos x$, $\tan^{-1}x$ or $\arctan x$, etc.

PRINCIPAL VALUES

For any angle, one, and only one, value of a trigonometric function corresponds to the angle; but for any value of a trigonometric function, numerous angles satisfy the value. Hence, the inverses of the trigonometric functions are not themselves functions. However, if we restrict the ranges of these relationships, we can obtain functions. The values of the trigonometric functions in the restricted ranges are called *principal values*. To indicate this restriction, we will capitalize the first letter in the name of the inverse trigonometric function; that is,

$$y = \mathrm{Sin}^{-1}x$$

or

$$y = \mathrm{Arcsin}\ x$$

Table 6-1.—Inverse Trigonometric Functions

FUNCTION	DOMAIN	RANGE
$y = \mathrm{Sin}^{-1}x$	$-1 \leq x \leq 1$	$-\pi/2 \leq y \leq \pi/2$
$y = \mathrm{Cos}^{-1}x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \mathrm{Tan}^{-1}x$	Any real number	$-\pi/2 < y < \pi/2$
$y = \mathrm{Cot}^{-1}x$	Any real number	$0 < y < \pi$
$y = \mathrm{Sec}^{-1}x$	$x \leq -1$ or $x \geq 1$	$0 \leq y \leq \pi$, $y \neq \pi/2$
$y = \mathrm{Csc}^{-1}x$	$x \leq -1$ or $x \geq 1$	$-\pi/2 \leq y \leq \pi/2$, $y \neq 0$

and so on for all the trigonometric functions. Table 6-1 shows the six inverse trigonometric functions, their domains, and their ranges.

EXAMPLE: Find all values of arctan 1.

SOLUTION: The tangent of many angles is 1, such as $\pi/4$, $5\pi/4$, $9\pi/4$, and $13\pi/4$. Thus the values of arctan 1 are

$$\frac{\pi}{4} + n\pi$$

where n is any integer.

EXAMPLE: Find Arctan 1.

SOLUTION: In the restricted range, as shown in table 6-1, the only number whose tangent is 1 is $\pi/4$. Hence,

$$\text{Arctan } 1 = \pi/4$$

EXAMPLE: Find Arcsec 2.236 in degrees.

SOLUTION: As previously determined,

$$\sec \theta = \frac{1}{\cos \theta}$$

If

$$x = \sec \theta$$

then

$$x = \frac{1}{\cos \theta}$$

or

$$\cos \theta = \frac{1}{x}$$

If we solve for θ in the above equations, then

$$\theta = \operatorname{arcsec} x$$

and

$$\theta = \arccos \frac{1}{x}$$

so,

$$\operatorname{arcsec} x = \arccos \frac{1}{x}$$

Hence, for the given problem

$$\operatorname{Arcsec} 2.236 = \operatorname{Arccos} \left(\frac{1}{2.236}\right)$$

$$= \operatorname{Arccos} 0.44723$$

According to appendix II, the angle whose cosine is 0.44723 is $63^\circ 26'$ to the nearest minute, which is in the range $0 \leq y \leq 180^\circ$. Therefore,

$$\operatorname{Arcsec} 2.236 = 63^\circ 26'$$

EXAMPLE: Find $\operatorname{Cos}^{-1}(-0.50000)$ in degrees.

SOLUTION: According to appendix II, the angle whose cosine is 0.50000 is 60°. However, we want the angle whose cosine is a negative number so that the angle is in the range of $0 \leq y \leq 180^\circ$. Since the cosine of a number is negative in the second quadrant where the reference angle of 60° corresponds to 120°, then

$$\operatorname{Cos}^{-1}(-0.50000) = 120^\circ$$

EXAMPLE: Find $\text{Cot}^{-1}(-\sqrt{3})$ in degrees.

SOLUTION: The expression $\text{Cot}^{-1}(-\sqrt{3})$ can be interpreted as "the angle between 0 and π, whose cotangent is $-\sqrt{3}$." Recall that

$$\cot \theta = \frac{\text{adjacent}}{\text{opposite}}$$

For the given problem,

$$\theta = \text{Cot}^{-1}(-\sqrt{3})$$

or

$$\text{Cot } \theta = -\sqrt{3}$$

From our previous discussion of special angles, you should recognize the reference angle of θ to be 30°. Since the cotangent of an angle is negative in the second quadrant for the range from 0° to 180°, then θ is 150°; that is,

$$\text{Cot}^{-1}(-\sqrt{3}) = 150°$$

EXAMPLE: Evaluate $\cos\left(\text{Arcsin } \frac{5}{13}\right)$.

SOLUTION: Let

$$u = \text{Arcsin } \frac{5}{13}$$

so

$$\text{Sin } u = \frac{5}{13}$$

Since Arcsin is defined only in quadrants I and IV and since 5/13 is positive, then u is in quadrant I. Figure 6-3 shows a triangle in quadrant I whose sine of angle u is 5/13. Using the Pythagorean theorem, we find that the side adjacent to angle u is 12. Therefore,

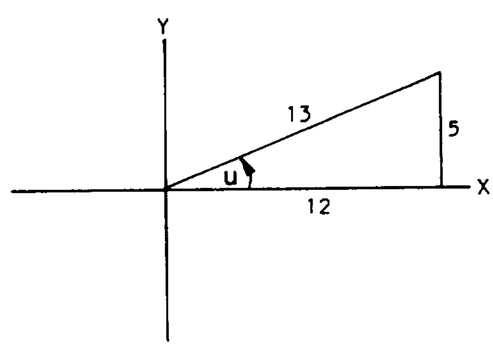

Figure 6-3.—Triangle in quadrant I.

$$\cos u = \frac{12}{13}$$

or

$$\cos\left(\text{Arcsin}\ \frac{5}{13}\right) = \frac{12}{13}$$

EXAMPLE: Evaluate $\sin\left(\text{Arccos}\ \frac{12}{13} - \text{Arcsin}\ \frac{4}{5}\right)$.

SOLUTION: Let

$$u = \text{Arccos}\ \frac{12}{13}$$

or

$$\text{Cos}\ u = \frac{12}{13}$$

and let

$$v = \text{Arcsin}\ \frac{4}{5}$$

or

$$\text{Sin}\ v = \frac{4}{5}$$

Angles u and v would both be in quadrant I according to previous conditions. Figure 6-4, view A, shows a triangle in quadrant I where $\cos u = 12/13$. Figure 6-4, view B, shows a triangle in quadrant I where $\sin v = 4/5$.

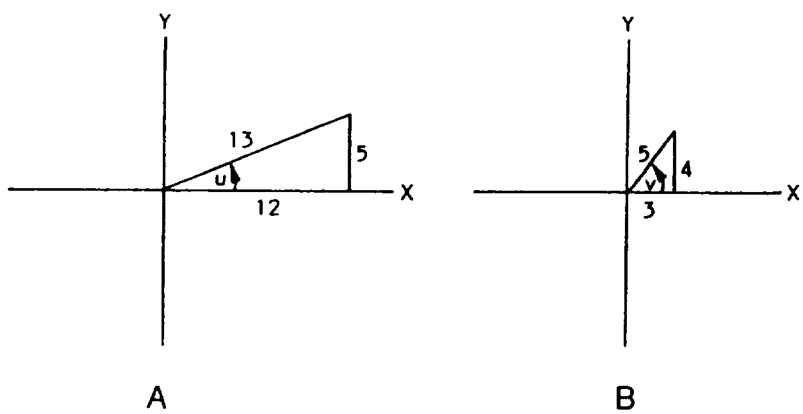

Figure 6-4.—Triangles in quadrant I.

Our given equation is in the form of the difference formula

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

where

$$\sin\left(\text{Arccos}\,\frac{12}{13} - \text{Arcsin}\,\frac{4}{5}\right)$$

$$= \sin\left(\text{Arccos}\,\frac{12}{13}\right)\cos\left(\text{Arcsin}\,\frac{4}{5}\right)$$

$$- \cos\left(\text{Arccos}\,\frac{12}{13}\right)\sin\left(\text{Arcsin}\,\frac{4}{5}\right)$$

Referring to figure 6-4, view A, we find that

$$\sin\left(\text{Arccos}\,\frac{12}{13}\right) = \frac{5}{13}$$

and

$$\cos\left(\text{Arccos}\,\frac{12}{13}\right) = \frac{12}{13}$$

Referring to figure 6-4, view B, we find that

$$\sin\left(\text{Arcsin}\,\frac{4}{5}\right) = \frac{4}{5}$$

and

$$\cos\left(\text{Arcsin}\,\frac{4}{5}\right) = \frac{3}{5}$$

Hence,

$$\sin\left(\text{Arccos}\,\frac{12}{13} - \text{Arcsin}\,\frac{4}{5}\right) = \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) - \left(\frac{12}{13}\right)\left(\frac{4}{5}\right)$$

$$= \frac{15 - 48}{65}$$

$$= -\frac{33}{65}$$

PRACTICE PROBLEMS:

1. Find $\mathrm{Cos}^{-1}(1/2)$.
2. Find Arcsin 0.88295 in degrees.
3. Find $\mathrm{Csc}^{-1}(-1.57208)$ in degrees.
4. Find Arccot $(-\sqrt{3}/3)$.
5. Evaluate tan [Arcsin $(-1/2)$].
6. Evaluate cot [$\mathrm{Cos}^{-1}(-0.19994)$].
7. Evaluate cos (Arctan 5/12 − Arccot 4/3).
8. Evaluate sin [$\mathrm{Sec}^{-1}(-25/24) - \mathrm{Csc}^{-1}(-17/8)$].

ANSWERS:

1. $\pi/3$
2. $62°$
3. $-39° 30'$
4. $2\pi/3$
5. $-1/\sqrt{3}$ or $-\sqrt{3}/3$
6. -0.20406
7. $63/65$
8. $-87/425$

TRIGONOMETRIC EQUATIONS

A *trigonometric equation* is an equality that is true for some values but may not be true for all values of the variable. The principles and processes used to solve algebraic equations may be used to solve trigonometric equations. The identities and formulas previously studied may also be used in solving trigonometric equations.

The following suggestions may be helpful to you in solving trigonometric equations:

1. *If only one trigonometric function is present, solve the equation for that function.*
2. *If more than one function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to zero to solve. You may find it helpful to use identities and formulas to change the form of the equation or to square both sides of the equation.*
3. *If the equation is quadratic in form, but not factorable, use the quadratic formula.*
4. *All possible solutions should be tested in the given equation.*

EXAMPLE: Solve $\tan \theta - 1 = 0$ for $0° \leq \theta < 360°$.

SOLUTION: We can rewrite

$$\tan \theta - 1 = 0$$

to read

$$\tan \theta = 1$$

or

$$\theta = \arctan 1$$

and solve the equation. Therefore, the solutions in the given interval are

$$\theta = 45° \text{ and } 225°$$

EXAMPLE: Solve $\sin 2\theta = 2 \cos \theta$ for $0° \leq \theta < 360°$.

SOLUTION: We will rearrange the equation so that one side equals 0; hence,

$$\sin 2\theta - 2 \cos \theta = 0$$

Since,

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

we will substitute this formula to change the form of our equation; that is,

$$2 \sin \theta \cos \theta - 2 \cos \theta = 0$$

We will now factor $2 \cos \theta$ from each term, so

$$2 \cos \theta (\sin \theta - 1) = 0$$

and set each factor equal to zero where

$$2 \cos \theta = 0$$

and

$$\sin \theta - 1 = 0$$

Solving each term gives

$$2 \cos \theta = 0$$

or

$$\theta = \arccos 0$$

where

$$\theta = 90° \text{ and } 270°$$

and

$$\sin \theta - 1 = 0$$

$$\sin \theta = 1$$

or

$$\theta = \arcsin 1$$

where

$$\theta = 90°$$

Substituting the value of $90°$ into the given equation gives

$$\sin 2(90°) = 2 \cos 90°$$

$$\sin 180° = (2)(0)$$

$$0 = 0$$

Substituting the value of $270°$ into the given equation gives

$$\sin 2(270°) = 2 \cos 270°$$

$$\sin 540° = 2(0)$$

$$0 = 0$$

Therefore, $\theta = 90°$ and $270°$ are the solutions to the equation.

EXAMPLE: Solve $\tan x - \sec x + 1 = 0$ for $0 \leq x < 2\pi$.

SOLUTION: Rewrite the given equation as

$$\tan x + 1 = \sec x$$

Square both sides of the equation to get

$$(\tan x + 1)^2 = (\sec x)^2$$

$$\tan^2 x + 2 \tan x + 1 = \sec^2 x$$

$$(\tan^2 x + 1) + 2 \tan x = \sec^2 x$$

Note that $\tan^2 x + 1 = \sec^2 x$, so

$$2 \tan x = 0$$

$$\tan x = 0$$

or

$$x = \arctan 0$$

Hence, the possible solutions are 0 and π. Substituting 0 into the original equation gives

$$\tan 0 + 1 = \sec 0$$

$$0 + 1 = 1$$

$$1 = 1$$

Substituting π into the original equation gives

$$\tan \pi + 1 = \sec \pi$$

$$0 + 1 = -1$$

but

$$1 \neq -1$$

Therefore, the only solution to the given equation is 0.

EXAMPLE: Solve $\cot^2\theta - 3 \cot \theta - 2 = 0$ for $0° \leq \theta < 360°$.

SOLUTION: Since this equation cannot be factored, we will use the quadratic formula, introduced in *Mathematics*, Volume 1,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the values for x are possible solutions to the equation

$$ax^2 + bx + c = 0$$

For our equation

$$x = \cot \theta$$

$$a = 1$$

$$b = -3$$

$$c = -2$$

Hence,

$$\cot \theta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{17}}{2}$$

So,

$$\cot \theta = 3.56155$$

or

$$\theta = \text{arccot } 3.56155$$

where

$$\theta = 15^\circ 41' \text{ and } 195^\circ 41'$$

to the nearest minute in quadrants I and III, respectively. And

$$\cot \theta = -0.56155$$

or

$$\theta = \operatorname{arccot}(-0.56155)$$

where

$$\theta = 119^\circ 19' \text{ and } 299^\circ 19'$$

to the nearest minute in quadrants II and IV, respectively. Substituting all four of the values of

$$\theta = 15^\circ 41', 119^\circ 19', 195^\circ 41', 299^\circ 19'$$

into the original equation shows that they are solutions.

NOTE: When substituting a possible solution into the original equation, we may not be able to equate the sides exactly because of rounding error.

PRACTICE PROBLEMS:

1. Solve $\sin \theta = -\sqrt{3}/2$ for $0^\circ \leq \theta < 360^\circ$.
2. Solve $\tan x \cos^2 x = \sin^2 x$ for $0 \leq x < 2\pi$.
3. Solve $\cot \theta - \csc \theta - \sqrt{3} = 0$ for $0^\circ \leq \theta < 360^\circ$.
4. Solve $7 \sin^2\theta - 3 \sin \theta - 4 = 0$ for $0^\circ \leq \theta < 360^\circ$.

ANSWERS:

1. $\theta = 240^\circ$ and 300°
2. $x = 0, \pi/4, \pi, 5\pi/4$
3. $\theta = 240^\circ$
4. $\theta = 90^\circ, 214^\circ 51'$, and $325^\circ 9'$

SUMMARY

The following are the major topics covered in this chapter:

1. **Suggestions in solving identities:**

    1. Know the basic identities.
    2. Attempt to transform the more complicated side into the other side.
    3. When possible, express all trigonometric functions in the equation in terms of sine and cosine.
    4. Perform any factoring or algebraic operations.

2. **Reciprocal identities:**

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

3. **Quotient identities:**

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

4. **Pythagorean identities:**

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

5. **Identities for negative angles:**

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

6. **Sum and difference formulas:**

The cosine of the difference of two angles is equal to the cosine of the first angle times the cosine of the second angle plus the sine of the first angle times the sine of the second angle; that is,

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

The cosine of the sum of two angles is equal to the cosine of the first angle times the cosine of the second angle minus the sine of the first angle times the sine of the second angle; that is,

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

The sine of the sum of two angles is equal to the sine of the first angle times the cosine of the second angle plus the cosine of the first angle times the sine of the second angle; that is,

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

The sine of the difference of two angles is equal to the sine of the first angle times the cosine of the second angle minus the cosine of the first angle times the sine of the second angle; that is,

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$$

The tangent of the sum of two angles is equal to the quantity of the tangent of the first angle plus the tangent of the second angle divided by the quantity of 1 minus the tangent of the first angle times the tangent of the second angle; that is,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

The tangent of the difference of two angles is equal to the quantity of the tangent of the first angle minus the tangent of the second angle divided by the quantity of 1 plus the tangent of the first angle times the tangent of the second angle; that is

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

7. **Double-angle formulas:**

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\begin{aligned} \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 1 - 2 \sin^2\alpha \\ &= 2 \cos^2\alpha - 1 \end{aligned}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2\alpha}$$

8. **Half-angle formulas:**

$$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$$

$$\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$$

$$\tan \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

9. **Inverse trigonometric functions:**

$x = \sin y$ or $y = \sin^{-1}x$ or $y = \arcsin x$

$x = \cos y$ or $y = \cos^{-1}x$ or $y = \arccos x$

$x = \tan y$ or $y = \tan^{-1}x$ or $y = \arctan x$

$x = \cot y$ or $y = \cot^{-1}x$ or $y = \operatorname{arccot} x$

$x = \sec y$ or $y = \sec^{-1}x$ or $y = \operatorname{arcsec} x$

$x = \csc y$ or $y = \csc^{-1}x$ or $y = \operatorname{arccsc} x$

10. **Principal values:** The values of the trigonometric functions in the restricted ranges are called *principal values.* This restriction is indicated by the capitalization of the first letter of the name of the inverse trigonometric function.

11. **Trigonometric equations:** A *trigonometric equation* is an equality that is true for some values but may not be true for all values of the variable.

12. **Suggestions in solving trigonometric equations:**

1. If only one trigonometric function is present, solve the equation for that function.
2. If more than one function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to zero to solve. You may find it helpful to use identities and formulas to change the form of the equation or to square both sides of the equation.
3. If the equation is quadratic in form, but not factorable, use the quadratic formula.
4. All possible solutions should be tested in the given equation.

ADDITIONAL PRACTICE PROBLEMS

1. Verify that $\frac{1}{1 - \cos x} + \frac{1}{1 + \cos x} = 2 \csc^2 x$.

2. Verify that $\frac{\sin (x + y)}{\cos (x - y)} = \frac{\cot x + \cot y}{1 + \cot x \cot y}$ using sum and difference formulas.

3. If $\tan \alpha = 8/15$ with α in quadrant I and $\cos \beta = 7/25$ with β in quadrant IV, find $\sec (\alpha - \beta)$.

4. Verify that $\tan x = \frac{1 - \cos 2x}{\sin 2x}$ using double-angle formulas.

5. Verify that $8 \sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right) = 1 - \cos 2x$ using half-angle formulas.

6. Find Arcsec (-2).

7. Find $\text{Tan}^{-1}(-0.12278)$ in degrees.

8. Evaluate $\sin [2 \text{ Tan}^{-1}(12/5)]$. HINT: $\sin 2\theta = 2 \sin \theta \cos \theta$.

9. Evaluate $\tan \left[\text{Arccos } \frac{\sqrt{3}}{2} - \text{Arcsin} \left(\frac{-3}{5}\right) \right]$.

10. Solve $\sin x + \cos x = \sqrt{2}$ if $0 \leq x < 2\pi$.

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. Result is known
2. Result is known
3. $-425/87$
4. Result is known
5. Result is known
6. $2\pi/3$
7. $-7°$
8. $120/169$
9. $\dfrac{4 + 3\sqrt{3}}{4\sqrt{3} - 3}$ or $\dfrac{25\sqrt{3} + 48}{39}$
10. $\pi/4$